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## MOTIONS IN HYPERSPACE.

BY C. L. E. MOORE.

1. The literature on motions in ordinary space is quite extensive but that on motions in hyperspace is very limited. The only paper that has a bearing on the subject in hand is one by Cole\* in which he studied rotations in 4-space. He found the invariant linear elements to be two completely perpendicular† planes (2-spaces). The method used is entirely unsuited to the study of motions in higher dimensions for the equations increase in complexity as the dimensions increase.

In this note I shall make use of Lie's infinitesimal transformations. Since every finite motion belongs to a one-parameter group, from the invariants of the one-parameter group the invariants of the general motion are obtained. Motions divide themselves into two classes according as the space is even or odd. The general motion in even space is found to be a rotation about a fixed point and leaves invariant  $n/2$  mutually completely perpendicular planes. If more than one point is left invariant, an even space will be left invariant. The general motion in odd space has an invariant line and the motion consists of rotations in the hyperplanes perpendicular to this line followed by a translation along the line. If a point is left invariant, an odd space is left fixed point for point.

In this discussion we shall consider only real motions.

2. **The group of infinitesimal transformations.** The group of motions in  $n$ -space is a group of  $\frac{1}{2}n(n + 1)$  parameters and consequently will be generated by  $\frac{1}{2}n(n + 1)$  independent infinitesimal transformations. The rotations in the coördinate planes (2-spaces) together with the translations along the  $n$  coördinate axes form a system of  $\frac{1}{2}n(n + 1)$  independent infinitesimal transformations. Let us denote this system by

$$(T) \quad \begin{aligned} X_i(f) &= \frac{\partial f}{\partial x_i}, \quad i = 1, 2, \dots, n, \\ X_{ij}(f) &= x_j \frac{\partial f}{\partial x_i} - x_i \frac{\partial f}{\partial x_j} = -X_{ji}, \quad i, j = 1, 2, \dots, n. \end{aligned}$$

We then have

$$\begin{aligned} (X_i X_j) &= X_i(X_j f) - X_j(X_i f) = 0, \\ (X_{ik} X_j) &= 0, \quad i, k \neq j, \quad (X_k X_{jk}) = X_j, \quad (X_j X_{jk}) = -X_k, \\ (X_{ij} X_{kl}) &= 0, \quad i, j \neq k, l, \quad (X_{ij} X_{ik}) = X_{jk}. \end{aligned}$$

\* On rotations in four dimensions, American Journal of Mathematics, vol. 12, 1889.

† Planes in which any line of one is perpendicular to every line of the other.

That is, the alternants of these transformations are all linearly expressible in terms of them and hence the transformations ( $T$ ) generate a group of  $\frac{1}{2}n(n + 1)$  independent parameters. These then will generate our group of motions in  $n$ -space.

The general infinitesimal transformation will be a linear combination of the transformations ( $T$ ).

$$X(f) = \Sigma a_i X_i + \Sigma a_{ij} X_{ij} = \Sigma_i (a_i + \Sigma_k a_{ik} x_k) \frac{\partial f}{\partial x_i},$$

where the  $a_{ij}$ 's satisfy the relation  $a_{ij} = -a_{ji}$ ,  $a_{jj} = 0$ . If this transformation leaves any points fixed

$$(1) \quad a_i + \Sigma_k a_{ik} x_k = 0, \quad i = 1, 2, \dots, n.$$

From the above relation on the coefficients  $a_{ij}$  we see that the determinant of (1) is skew-symmetric. The rank of a skew-symmetric determinant being even, it is at once seen that the solution of (1) depends first of all on whether  $n$  is odd or even.

**3. General motion in even space.** If  $n = 2m$ , say, the determinant  $|a_{ik}|$  in general does not vanish and (1) has a solution, hence one point is left invariant. If this invariant point be taken for the origin the transformations of the group become

$$X = \Sigma a_{ij} X_{ij},$$

and equation (1) becomes

$$\Sigma a_{ik} x_k = 0.$$

If  $|a_{ik}|$  is of rank  $2m$  the origin is the only invariant point. If the rank is  $2m - 2$  there is a plane of invariant points and in general a linear space of even dimensions is always left invariant. When the rank is 2 a space of  $2m - 2$  dimensions is left invariant point for point. In this case the motion is a rotation parallel to a fixed plane and is therefore simply isomorphic with the rotations in a plane.

In the general case only a single point is left fixed. We shall now examine whether or not there are any linear spaces which are left invariant as a whole (the individual points not being left invariant). Let us see first if any hyperplanes passing through the fixed point  $O$  are left invariant. Since the origin is left fixed the transformation belongs to the group of homogeneous transformations. The equation of a hyperplane passing through  $O$  is

$$\Sigma \lambda_k x_k = 0,$$

and the condition that this be left invariant is

$$\Sigma_i \Sigma_k \lambda_i a_{ik} x_k = \rho \Sigma_k \lambda_k x_k,$$

that is

$$(2) \quad \Sigma_i \lambda_i a_{ik} = \rho \lambda_k.$$

This being homogeneous in  $\lambda$  will have a solution only when the determinant of the system vanishes. This determinant is

$$\begin{vmatrix} -\rho & a_{12} & a_{13} & \cdots & a_{1, 2m} \\ -a_{12} & -\rho & a_{23} & \cdots & a_{2, 2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -a_{1, 2m} & \cdot & \cdot & \cdot & -\rho \end{vmatrix}$$

and is seen to be a skew determinant. If such a determinant is expanded in powers of the diagonal terms, the coefficients of the various powers of  $\rho$  will be sums of principal minors of  $|a_{ik}|$ . That is, in the equation

$$(3) \quad \rho^{2m} + A_1\rho^{2m-1} + A_2\rho^{2m-2} + \cdots A_n = 0$$

$A_1 = 0$ ,  $A_2$  is the sum of all the principal minors of  $|a_{ik}|$  of order two (that is, minors whose principal diagonals are elements of the principal diagonal of  $|a_{ik}|$ ),  $A_3$  is the sum of all the principal minors of order three, etc. Thus all the coefficients are sums of skew-symmetric determinants. But such determinants of odd order vanish and those of even order are perfect squares. Hence in (3) there are no odd powers of  $\rho$  and the coefficients of the even powers are all positive. Therefore (3) has  $m$  pairs of conjugate imaginary roots. *There are then  $m$  pairs of conjugate imaginary hyperplanes left invariant by a general real motion in a space of  $2m$  dimensions.* A pair of conjugate imaginary hyperplanes will intersect in a real space of  $2m - 2$  dimensions, a pair of these will intersect in a real space of  $2m - 4$  dimensions and finally  $m - 1$  of these will intersect in a real plane. Hence a motion in a space of  $2m$  dimensions leaves  $m$   $(2m - 2)$ -dimensional spaces invariant,  $\frac{1}{2}m(m - 1)$  spaces of  $2m - 4$  dimensions, and finally  $m$  planes invariant. All these spaces pass through the origin and are real.

If a plane is left invariant, then the  $S_{2m-2}$  perpendicular to it and passing through  $O$  must also be left invariant. For the  $S_{2m-2}$  is generated by the lines passing through  $O$  and perpendicular to the fixed plane and since the origin is left fixed this set of lines will go into itself and hence the space generated by them is left invariant. In this  $S_{2m-2}$  a plane is left invariant and these two planes are completely perpendicular. In the  $S_{2m-4}$  lying the  $S_{2m-2}$  and perpendicular to the second plane, a plane is also invariant which is completely perpendicular to the other two and so on. Hence *the general motion in  $2m$  dimensions leaves invariant  $m$  completely perpendicular planes.*

There are  $2m$  minimal hyperplanes left invariant. For the minimal cone  $\Sigma x_i^2 = 0$  is left invariant and hence tangent planes to it go into tangent planes. Through each invariant  $S_{2m-2}$  pass two of these tangent hyper-

planes and as the motion is an infinitesimal one these must be left invariant. Hence the  $2m$  invariant hyperplanes must be minimal hyperplanes.

If we take these invariant planes as the coördinate planes the transformation ( $T$ ) takes the form

$$x_2 \frac{\partial f}{\partial x_1} - x_1 \frac{\partial f}{\partial x_2}, \quad x_4 \frac{\partial f}{\partial x_3} - x_3 \frac{\partial f}{\partial x_4}, \quad x_6 \frac{\partial f}{\partial x_5} - x_5 \frac{\partial f}{\partial x_6}, \quad \dots, \\ x_{2m} \frac{\partial f}{\partial x_{2m-1}} - x_{2m-1} \frac{\partial f}{\partial x_{2m}}.$$

Taking this form for ( $T$ ) we see that the path curves of the group are the solutions of the set of equations

$$a_1 \frac{dx_1}{x_2} = -a_1 \frac{dx_2}{x_1} = a_2 \frac{dx_3}{x_4} = -a_2 \frac{dx_4}{x_3} = \dots = a_m \frac{dx_{2m-1}}{x_{2m}} = -a_m \frac{dx_{2m}}{x_{2m-1}}.$$

Hence the path curves lie on the intersection of the cylinders

$$x_1^2 + x_2^2 = \bar{x}_1^2 + \bar{x}_2^2, \quad x_3^2 + x_4^2 = \bar{x}_3^2 + \bar{x}_4^2, \quad \dots, \\ x_{2m-1}^2 + x_{2m}^2 = \bar{x}_{2m-1}^2 + \bar{x}_{2m}^2,$$

where  $\bar{x}_1, \bar{x}_2, \bar{x}_3, \dots, \bar{x}_{2m}$  are the initial coördinates of the point. Hence we see that the projection of the path curves on each of the invariant planes is a circle. This fact is evident geometrically for if the point  $P$  (not in a fixed plane) projects on one of the invariant planes into the point  $Q$ , then the triangle  $OPQ$  by the group will be so moved that  $O$  remains fixed, while  $Q$  will always remain in the fixed plane. The distance  $OQ$  is left invariant and hence  $Q$  will describe a circle. The locus of  $P$  is not a circle, for this would mean that the plane of the circle is invariant, which, in general, is impossible.

If  $|a_{ik}| = 0$ , we see from equation (3) that two values of  $\rho$  are zero. In this case it is seen from (2) that a one-parameter family of real hyperplanes is left invariant, since the rank of  $|a_{ik}|$  is now  $2m - 2$ . We saw that this was the condition for a plane of invariant points. The invariant hyperplanes are then those containing the  $S_{2m-2}$  perpendicular to this absolutely invariant plane. This is evident since in this case the same equations that determine the coördinates of the hyperplane also determine the coördinates of the invariant points. The coördinates of any point  $P$  in the fixed plane are proportional to the direction cosines of the hyperplane perpendicular to  $OP$ . Now if these coördinates are the same as the coördinates of one of these hyperplanes then this hyperplane must be perpendicular to  $OP$ . Hence the invariant hyperplanes contain the  $S_{2m-2}$  normal to the fixed plane.

In this case there are still  $m$  invariant planes but one of them is left absolutely invariant. In the same way we see that if the rank of  $|a_{ik}|$  is  $2m - 4$ , there is a three-parameter family of real invariant hyperplanes, and two of the invariant planes are point for point invariant. The same reasoning holds when the rank of  $|a_{ik}|$  is reduced two at a time. If the rank is less than 2 the motion reduces to an identity. If  $m - 1$  of the fixed planes are absolutely invariant ( $X$ ), becomes the group of motions in a plane; if  $m - 2$  are absolutely invariant, the group becomes the rotations in  $S_4$  and so on.

**4. Motions in even space which have no fixed points.** We have seen that, in general, in a space of even dimensions one point at least is always left fixed. We know, however, that in the plane there are motions which have no fixed points, viz., translations. We shall now examine what are the conditions that the same thing happen in higher dimensions. Equation (1), if  $|a_{ik}| = 0$ , are inconsistent except for particular values of  $a_i$ . In case they are inconsistent we may say that the invariant point has moved off to infinity. This would then indicate that there is an invariant plane in which a pencil of parallel lines and consequently every pencil of parallel lines are left invariant. We will now show that this is the case. There are  $\infty^1$  directions left invariant. For the coefficients of direction of the line joining two points  $x, y$  are  $x_i - y_i$  and if we write the equations of the motion in the form

$$(4) \quad x_i' = x_i + (a_i + \Sigma a_{ij}x_j)dt$$

we have at once the relation

$$x_i' - y_i' = x_i - y_i + \Sigma a_{ij}(x_j - y_j)dt.$$

The direction  $x_j - y_j$  will then be left invariant if

$$(5) \quad \Sigma a_{ij}(x_j - y_j) = 0.$$

But since we assumed that  $|a_{ij}| = 0$  and consequently its first minors, it follows that there are at least  $\infty^1$  sets of values of  $x_j - y_j$ , not all zero, which will satisfy (5) and which are therefore left invariant by the motion.

We can now show that there is one pencil of these lines in which the individual lines are left invariant, that is, the points of one of those lines move along the line itself. A plane containing two of these invariant directions  $u_i, u_i'$  can be written

$$(6) \quad x_i = \bar{x}_i + u_i d\xi + u_i' d\eta.$$

Since  $u_i, u_i'$  are homogeneous we can choose them so that they will satisfy the relations

$$\Sigma u_i^2 = 1, \quad \Sigma u_i'^2 = 1, \quad \Sigma u_i u_i' = 0.$$

The first two say that the  $u$ 's are actual direction cosines and the third that the two directions are perpendicular. Now by (4)  $\bar{x}_i$  is transformed into  $\bar{x}_i + (a_i + \Sigma a_{ij}\bar{x}_j)dt$  and if this new point satisfies (6) we must be able to find values of  $d\xi/dt$ ,  $d\eta/dt$  which will render the system of equations

$$(7) \quad \Sigma a_{ij}x_j + a_i = u_i \frac{d\xi}{dt} + u'_i \frac{d\eta}{dt}$$

consistent. The determinant  $|a_{ij}|$  of this system of equations is of rank  $2m - 2$  or lower, and therefore if they are consistent there must be two linear relations connecting them. Multiplying by  $u_i$  and summing on  $i$  and then on  $u'_i$  and summing on  $i$ , we have

$$\Sigma a_i u_i = \frac{d\xi}{dt}, \quad \Sigma a_i u'_i = \frac{d\eta}{dt}$$

by virtue of (5) and the relation between the  $u$ 's. We then have a set of values of  $d\xi/dt$  and  $d\eta/dt$  which will render the set of equations (7) consistent. The values of  $\bar{x}_i$  will then be the solutions of (7) after we have replaced  $d\xi/dt$ ,  $d\eta/dt$  by the values found above. But in (7) there are only  $2m - 2$  independent equations and therefore there is a plane of these solutions. Hence we have a plane which is invariant and which contains a pencil of invariant parallel lines. In this plane any line is transformed into a parallel line and consequently in it the transformation is a translation.

An  $S_{2m-2}$  normal to the plane just found will cut it in a single point. By (6) this point is translated along the invariant line passing through it and the  $S_{2m-2}$  itself is rotated parallel to itself. This motion then is simply isomorphic to the general motion in an  $S_{2m-1}$  determined by the  $S_{2m-2}$  and the invariant line cutting it. The motions in odd space will be taken up later. If we take the origin in this plane and the  $x_{2m-1}$ -axis as one of the invariant lines and the line in the plane perpendicular to it as the  $x_{2m}$ -axis, then the transformations of such a motion can be written

$$x_2 \frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}, x_4 \frac{\partial f}{\partial x_3} - x_3 \frac{\partial f}{\partial x_4}, \dots, x_{2m-3} \frac{\partial f}{\partial x_{2m-2}} - x_{2m-2} \frac{\partial f}{\partial x_{2m-3}}, \frac{\partial f}{\partial x_{2m-1}}.$$

We shall see later that these are identical with the equations for the general motion in odd space.

The equation of a hyperplane which passes through the fixed plane is

$$\sum_1^{2m-2} \lambda_i x_i = 0.$$

If this hyperplane is left invariant by (4), it would be necessary that it be

left invariant by the rotation

$$x'_i = x_i + \sum a_{ij} x_j dt, \quad i, j = 1, 2, \dots, 2m-2.$$

This then becomes identical with the problem previously solved, that is, that represented by equation (2). There are then  $m-1$  pairs of conjugate imaginary hyperplanes left invariant by (4). By reasoning similar to that used before we see that there are  $m-1$  spaces of four dimensions left invariant. These are the four-spaces determined by the invariant plane and any one of the  $m-1$  planes left invariant in an  $S_{2m-2}$  normal to the invariant plane by the rotation part of the group. That is, we consider the motion to be broken up into a rotation followed by a translation. Within one of these invariant four-spaces there is also an infinite number of three-spaces left invariant. They will be those determined by an invariant plane in the  $S_{2m-2}$  and the invariant line which cuts it.

It is easy to see what the invariant configurations are in case  $|a_{ij}|$  is of lower rank.

**5. Motions in odd space.** If  $n$  is odd the condition for invariant points is again given by equation (1). But in this case the determinant being skew-symmetric of odd order must vanish and hence (1) is an inconsistent set of equations except for particular values of  $a_{ij}$ . By processes similar to those previously used we see that the condition that the direction of a line joining two points  $x, y$  be invariant is

$$\sum a_{ij} u_j = 0, \quad i = 1, 2, \dots, n,$$

where  $u_i = x_i - y_i$ . Since  $|a_{ij}| = 0$ , it follows that there is always one invariant direction. Since the  $u$ 's are a homogeneous set, we can determine them so that  $\sum u_i^2 = 1$ . This of course excludes minimal lines but since we are interested only in real motions the coefficients  $a_{ij}$  are real and consequently the  $u$ 's are real and the minimal lines are excluded.

If the rank of  $|a_{ij}|$  is  $n-1$ , there is just one invariant line. To show this we proceed the same as for the case when  $n$  was even. A line having this direction may be written

$$x_i = \bar{x}_i + u_i d\tau.$$

Now by (1)  $\bar{x}_i$  is transformed into  $\bar{x}_i + (a_i + \sum a_{ij} \bar{x}_j) dt$  and, if this point lies on the line, we have

$$a_i + \sum a_{ij} \bar{x}_j = u_i \frac{d\tau}{dt}.$$

Now it must be possible to find a value of  $\tau$  which will render this system

consistent. This will be so when a linear combination of them vanishes. Multiply each equation by  $u_i$  and sum, then

$$\tau = \sum a_i u_i.$$

Substituting this value of  $\tau$ , we have the invariant line determined as the intersection of the hyperplanes

$$\sum a_{ij} x_j + a_i = u_i \frac{d\tau}{dt}.$$

There are  $\infty^1$  solutions of this set of equations since the determinant is of rank  $n - 1$ .

If the values of  $a_i$  are such as to render equations (1) consistent, then there will be a line of invariant points, that is the invariant line has become fixed point for point. In this case the motion reduces to a rotation about the fixed line and is simply isomorphic to the rotations in the even space  $S_{n-1}$  perpendicular to the axis of rotation. We may say then if  $|a_{ij}|$  is of rank  $n - 1$ , there is either no fixed point or a line of fixed points.

If the rank of  $|a_{ij}|$  reduces to  $n - 3$ , then by continuing the same argument as above we see that there are  $\infty^2$  directions left invariant and that  $\infty^2$  lines of an  $S_3$  having this direction are left invariant. The motion then consists of a rotation in an  $S_{n-3}$  perpendicular to the  $S_3$  followed by a translation along the invariant line which cuts it. In this case if  $a_i$  are such as to render the system of equations consistent, then the  $S_3$  will be left invariant point for point. The other cases follow the same arguments. We may then say in general, if no point is left invariant, then an odd space generated by a system of parallel invariant lines is left invariant. If one point is left invariant, then an odd space is left absolutely invariant.

In the general case if the axis  $x_n$  is taken as the fixed line, the infinitesimal transformations of the group become

$$x_i' = x_i + \sum a_{ij} x_j dt, \quad i, j = 1, 2, \dots, n - 1,$$

$$x_n' = x_n + a_n dt.$$

The equation of a hyperplane which contains the invariant line is

$$\sum_i^{n-1} \lambda_i x_i = 0,$$

and if it is left invariant,

$$\sum \sum \lambda_i a_{ij} x_j = \rho \sum \lambda_i x_i.$$

This is the same as equation (2) and hence there are  $(n - 1)/2$  invariant conjugate imaginary hyperplanes and consequently  $(n - 1)/2$  real invariant  $S_3$ 's. Again this could have been inferred from the fact that in an  $S_{n-1}$  perpendicular to  $x_n$  there are  $(n - 1)/2$  invariant planes and an  $S_3$  determined by each of these planes and  $x_n$  is left invariant. The cases in which the rank of  $|a_{ij}|$  is less than  $n - 1$  are easily handled as before.

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